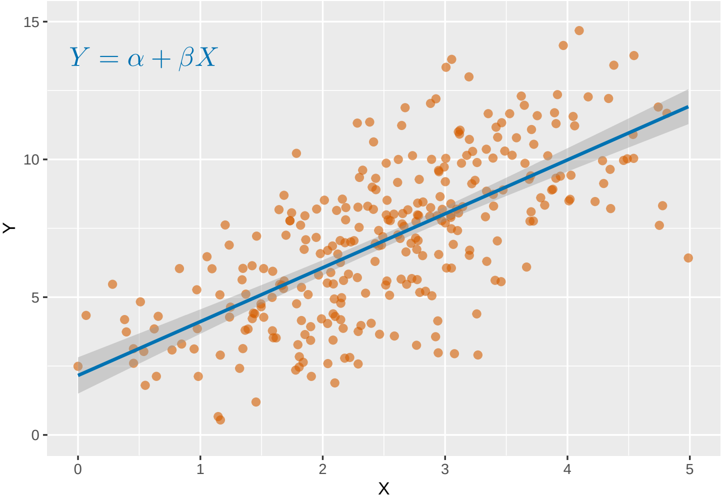
Williams College ECON 370:

Data Science for Economic Analysis

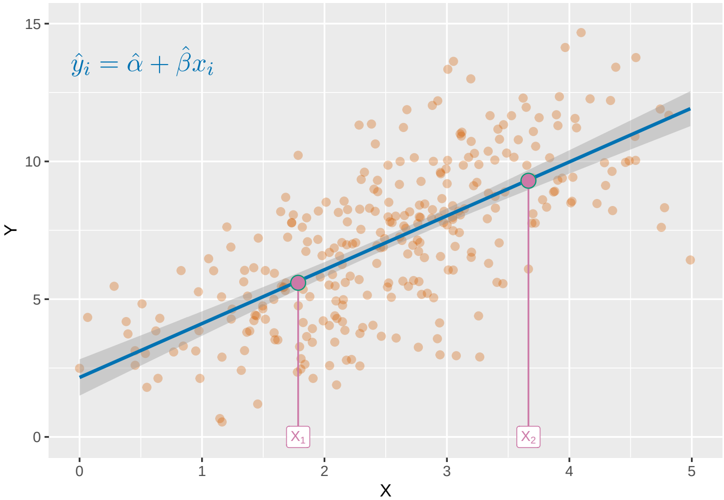
Lecture 4: Numerical Approaches to OLS

Professor: Pamela Jakiela

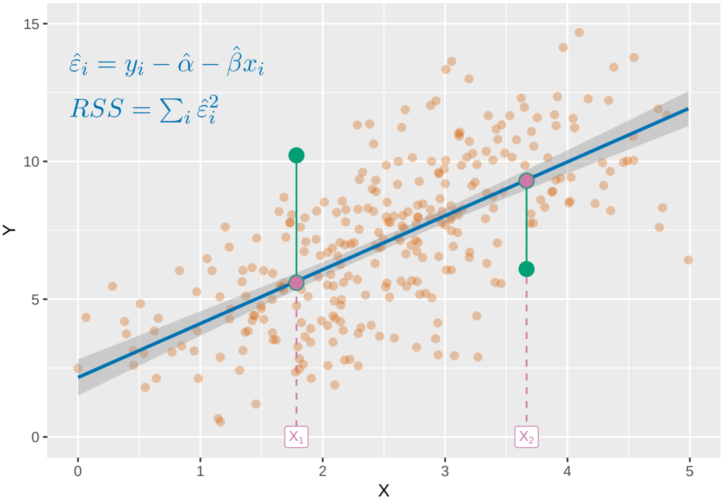
There's Nothing Ordinary About OLS



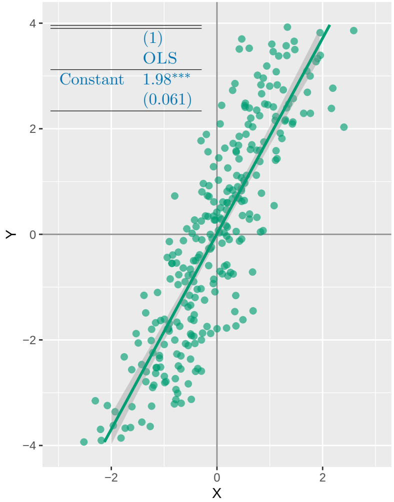
OLS Generates Predicted Values of Y



OLS Coefficients Minimize the Sum of Squared Residuals



$\hat{\beta}_{ols}$ Minimizes the Residual Sum of Squares (RSS)



$$RSS = \sum_i \varepsilon_i^2 = \sum_i (y_i - \hat{\beta} x_i)^2$$

$$\Leftrightarrow \frac{\partial RSS}{\partial \beta} = \sum_i 2(y_i - \hat{\beta} x_i) x_i = \sum_i 2 y_i x_i - \sum_i 2 \hat{\beta} x_i^2$$

Solving the first-order-condition for $\hat{\beta}$:

$$\frac{\partial RSS}{\partial \beta} = 0 \Leftrightarrow \hat{\beta}_{ols} = \sum_i y_i x_i / \sum_i x_i^2$$

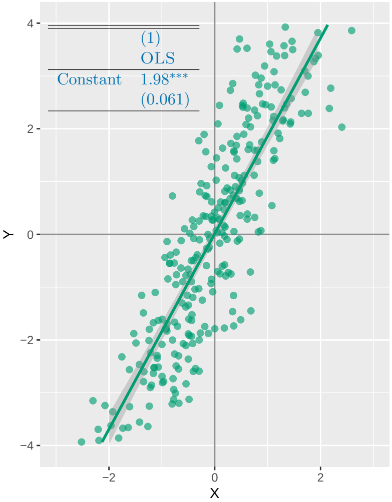
Generalizes to bivariate OLS with a constant:

$$\hat{\beta}_{ols} = \frac{COV(X,Y)}{VAR(X)} = \sum_i y_i (x_i - \bar{x}) / \sum_i (x_i - \bar{x})^2$$

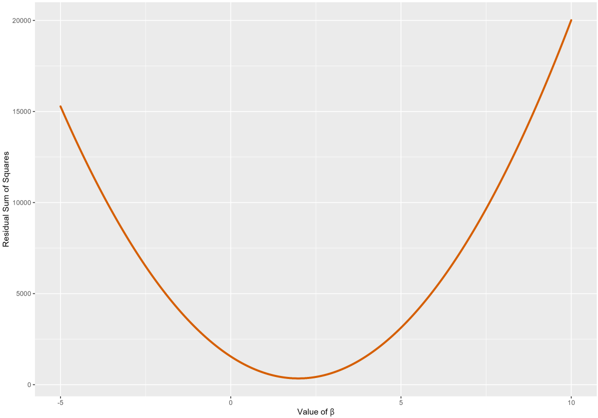
Generalizes to multivariate regression case:

$$\hat{\beta}_{ols} = (X'X)^{-1} X'Y$$

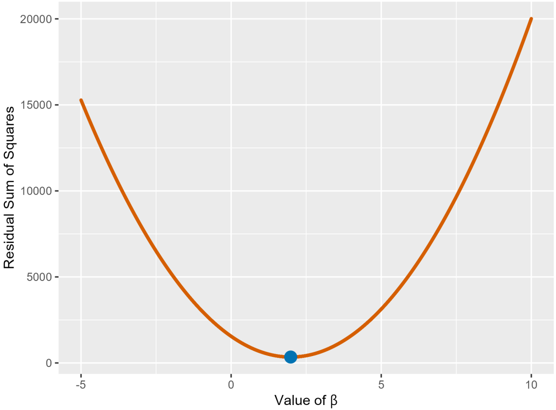
A Numerical Approach to OLS: Find the β that Minimizes RSS



$$RSS(\beta) = \sum_i [\varepsilon_i(\beta)]^2 = \sum_i [y_i - (\beta)x_i]^2$$



Grid Search

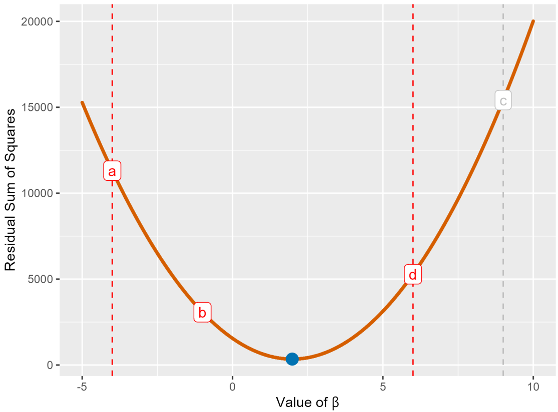


Simplest approach to choosing β: calculate $RSS(\beta)$ for a range of βs (sufficiently widow search window), choose β that yields lowest RSS

Grid search is computationally costly, not feasible in many dimensions

Grid search can still be useful, especially when objective function is not smooth and/or convex or parameters are near the boundaries

Golden Search



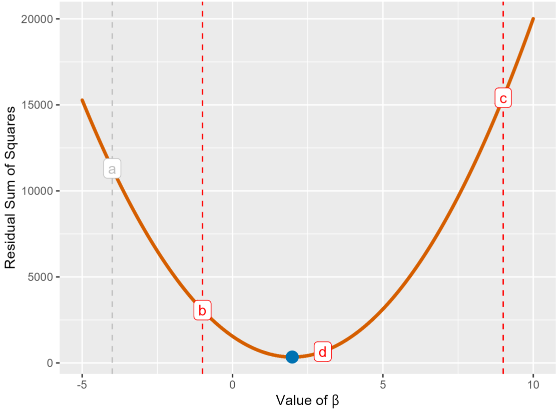
Suppose $a < b < c$, but
$f(b) < f(a)$ and $f(b) < f(c)$

$\Rightarrow$ minimum is between a and c

Choose d between b and c

If $f(d) > f(b)$, then
the minimum is between a and d

Golden Search



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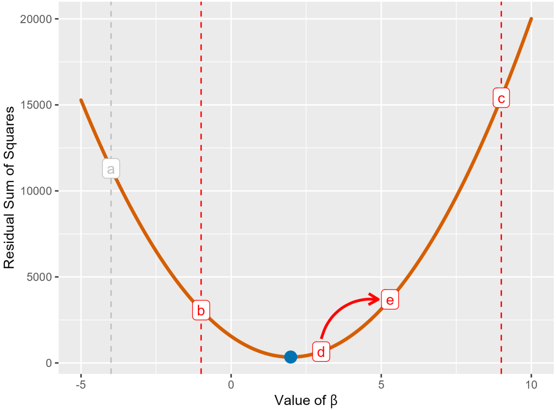
Golden Search

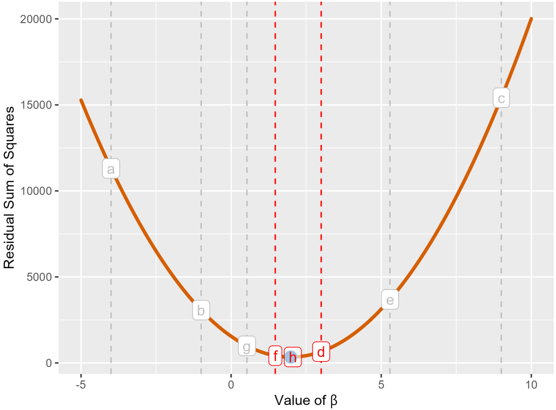


Why is it called Golden Search?

Optimal step size as a fraction of distance from midpoint to far end:

$\frac{3-\sqrt{5}}{2} \approx 0.382$ ("the golden mean")

Golden Search



Why is it called Golden Search?

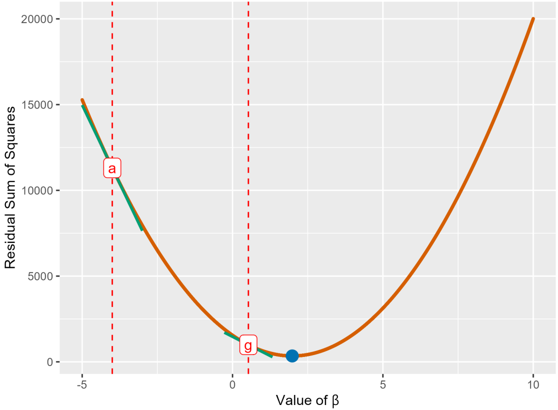
Optimal step size as a fraction of distance from midpoint to far end:

$\frac{3-\sqrt{5}}{2} \approx 0.382$ ("the golden mean")

Stopping rule: when $f(x)$ converges

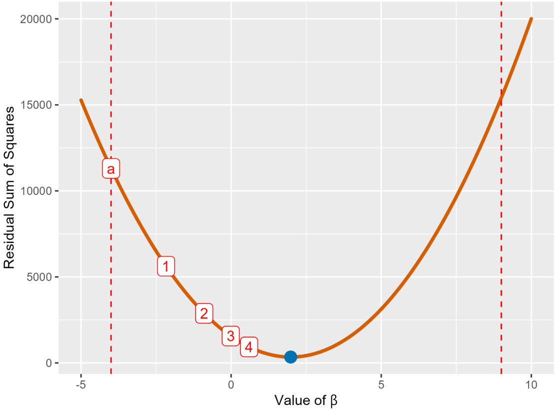
Here, **convergence** means that the optimal β is bounded in a sufficiently narrow window

Gradient Descent



Use the slope (i.e. the **gradient**) to to decide direction to go and how far

Gradient Descent



Use the slope (i.e. the **gradient**) to to decide direction to go and how far

$$x_1 = x_0 - \alpha \left[f'(x_0) \right]$$

Steps gets smaller and smaller as you approach the minimum of $RSS(\beta)$

Importantly: generalizes to vector βs

Numerical Minimization in Multiple Dimensions

In multiple dimensions, algorithms define direction and step size, and convergence occurs when subsequent values of the function being optimized are sufficiently close together (tolerance)

- **Gradient descent:** from an arbitrary starting point, steps in parameter space are proportional to the gradient of the function (i.e. the vector of partial derivatives)
  - Can be slow because it sometimes requires a large number of steps to reach convergence
- **Newton's method:** from an arbitrary starting point, steps are minus one times the ratio of the gradient to the Hessian (i.e. the matrix of second derivatives including cross partials)
  - Can be slow because of the time required to calculate the Hessian
- **Broyden-Fletcher-Goldfarb-Shanno (BFGS):** a quasi-Newton method, similar to Newton's method but using a less computationally-intensive approximation of the Hessian

Summary

- OLS coefficients are chosen to minimize the RSS
- OLS can be framed as a numerical minimization problem
- We can find OLS coefficients using standard tools for numerical optimization
- Aside: OLS also solves the maximum likelihood estimation problem with normal residuals

Lab #4

The objective of this lab is to familiarize ourselves with: simulating data-generating processes, the mechanics of linear regression (again!), defining functions, and numerical optimization

1. Simulate a data set
2. Run OLS
3. Calculate OLS coefficients "by hand" (using the formula) in the bivariate, no-constant case
4. Write a function to calculate the RSS and find $\hat{\beta}_{ols}$ through grid search
5. Find $\hat{\beta}_{ols}$ through numerical optimization by finding the minimum of $RSS(\beta)$

Numerical Minimization in Practice: Pseudo-Random Numbers

R:

```
set.seed(8675309)
A <- matrix(rnorm(4*3), 4, 3)
```

$$A = \begin{bmatrix} -0.996 & 1.065 & 0.572 \\ 0.721 & 0.987 & 0.903 \\ -0.617 & 0.027 & -1.549 \\ 2.029 & 0.672 & 1.022 \end{bmatrix}$$

Python:

```
import numpy as np
np.random.seed(8675309)
B = np.random.randn(4, 3)
```

$$B = \begin{bmatrix} 0.589 & 0.733 & -1.162 \\ -0.556 & -0.772 & -0.168 \\ -0.416 & -1.378 & 0.749 \\ 0.178 & 0.694 & -1.978 \end{bmatrix}$$

Numerical Minimization in Practice: Functions (RSS)

R:

```
RSS <- function(beta){
    yhat <- X %*% beta
    return(sum((Y - yhat)^2))
}
```

Python:

```
def RSS(beta):
    yhat = X @ beta
    return np.sum((Y - yhat) ** 2)
```

`%*%` and `@` indicate matrix multiplication:

$$\underset{1 \times 3}{\begin{bmatrix} a & b & c \end{bmatrix}} \underset{3 \times 1}{\begin{bmatrix} d \\ e \\ f \end{bmatrix}} = \underset{1 \times 1}{\begin{bmatrix} ad + be + cf \end{bmatrix}}$$

$$\underset{\color{red}{2 \times 3}}{\begin{bmatrix} a & b & c \\ A & B & C \end{bmatrix}} \underset{\color{red}{3 \times 1}}{\begin{bmatrix} d \\ e \\ f \end{bmatrix}} = \underset{\color{red}{2 \times 1}}{\begin{bmatrix} ad + be + cf \\ Ad + Be + Cf \end{bmatrix}}$$

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`%*%` and `@` indicate matrix multiplication:

$$\underbrace{\begin{bmatrix} 1 & x_{11} & x_{12} \\ 1 & x_{21} & x_{22} \\ 1 & x_{31} & x_{32} \\ 1 & x_{41} & x_{42} \end{bmatrix}}_{4 \times 3} \underbrace{\begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix}}_{3 \times 1} = \begin{bmatrix} \beta_0 + \beta_1 x_{11} + \beta_2 x_{12} \\ \beta_0 + \beta_1 x_{21} + \beta_2 x_{22} \\ \beta_0 + \beta_1 x_{31} + \beta_2 x_{32} \\ \beta_0 + \beta_1 x_{41} + \beta_2 x_{42} \end{bmatrix} = \underbrace{\begin{bmatrix} \hat{y}_1 \\ \hat{y}_2 \\ \hat{y}_3 \\ \hat{y}_4 \end{bmatrix}}_{4 \times 1}$$

Numerical Minimization in Practice: Numerical Optimization

R:

```
b0 <- rep(0, numvars + 1)
result <- optim( b0, RSS, method = "BFGS" control = list(maxit = 1e5) )
```

numerical optimization function

function to minimize

starting values of parameters

method

maximum number of iterations

Python:

```
from scipy.optimize import minimize
b0 = np.zeros(numvars + 1)
result = minimize( RSS, b0, method='BFGS', options='maxiter': int(1e5) )
```