10 Production

10.1 Production Functions

- A **producer** is a person or organization who transforms inputs (such as capital, raw materials, or labor) into outputs using a production technology.
- A producer is characterized by a **production function** $q = f(z)$ where q indicates the quantity of output produced and z indicates the quantity of a single input good used up producing the output.
- We make the following assumptions about production functions:
  - $f'(z) > 0$ (production functions are increasing)
  - $f''(z) < 0$ (production functions are concave)
  - $f(0) = 0$ (no free lunch: zero inputs means zero output)
- Concavity means that for all positive z and a,

  $$f(z) - f(z-a) > f(z+a) - f(z) \tag{1}$$

  In other words, it gets harder and harder to increase output by using up more inputs.
- There are a few different ways to justify (or just think about) this assumption of concavity:
  - Production often involves more than one input. Modeling production as a function of a single input often makes sense when only one input is variable in the short run.
  - We don't (typically) see producers scaling up production until they produce an infinite amount of output, so the production function must turn concave eventually
  - Our model of labor supply covers cases where a producer's output is linear in their effort

10.2 The Producer's Objectives

- What is the producer's objective? Profit maximization? Selling as much of their product as possible without going out of business? Employing as many people as possible? Maximizing the price of company stock?

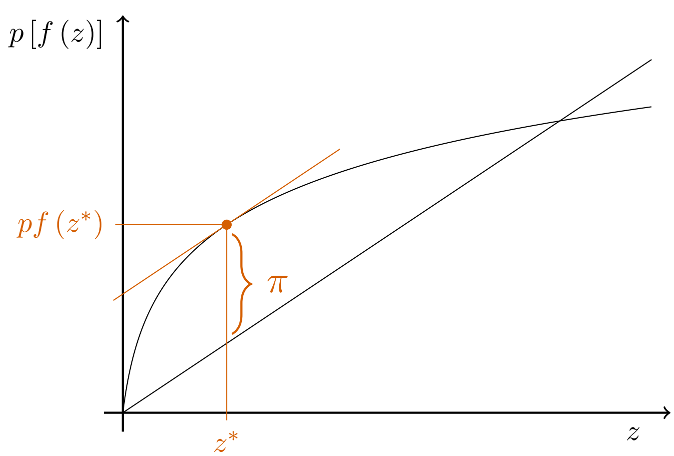
- When a producer's choices do not impact input or output prices, we can write the producer's profit function as:

$$\pi(z) = p\,[f(z)] - wz \tag{2}$$

  where $p > 0$ indicates the price that the producer is paid for each unit of output produced and $w > 0$ indicates the price that the producer must pay for each unit of input used up in production. $\pi(z)$ are the producer's profits.

- $p\,[f(z)]$ represents **revenues** – money earned from the sale of output. Because the production function is concave, $p\,[f(z)]$ is also concave.

- Because the per-unit cost of the input good, w, does not depend on z, the producer's costs increase linearly with the quantity of inputs used in production.



- At low levels of output, revenues are increasing faster than input costs because the production function is concave, but at high levels of production the opposite is true

- A producer maximizes profits by choosing the level of output where the slope of the revenue curve is equal to the slope of the marginal input cost, w

- We can identify this output level by taking the derivate of the profit function with respect to z:

$$\max_{z \geq 0} \pi(z) = p\left[f(z)\right] - wz \Rightarrow p\left[f'(z^*)\right] = w \tag{3}$$

- **Example:** production function $f(z) = z^{\frac{1}{3}}$
- **Question.** How would the example be different if your firm were producing the output, and the only input was your labor (which you didn't have to pay for)?
  - **Answer.** It wouldn't: if you could sell your labor for wage rate w, the opportunity cost of using your labor to produce output is still w. Implicitly, if the firm can use your labor without paying you a wage, it could also rent out your labor to another firm and include your labor income as part of its profit.
  - Of course, the answer would be different if you were unemployed and couldn't find work (see: Arthur Lewis' dual-sector model for the implications of this)

10.3 Cost Functions

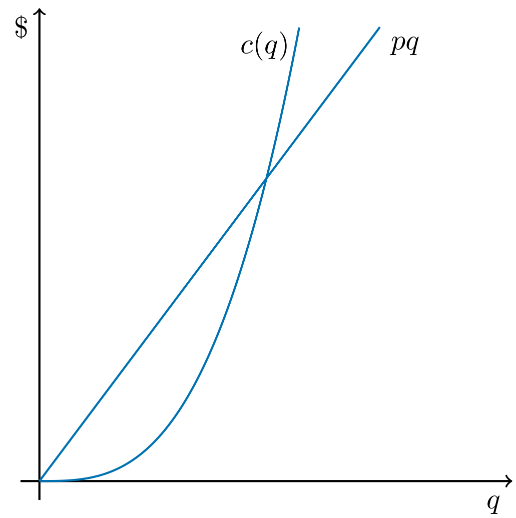
- Instead of thinking about how much output we can produce with z units of the input good, we can instead frame the producer's problem in terms of choosing a level of output q and determining how many units of the input good are needed to produce q

$$q = f(z) \Leftrightarrow z = f^{-1}(q) \tag{4}$$

- The producer's **cost function** tells us the cost of producing q units of output:

$$c(q) = w\left[f^{-1}(q)\right] \tag{5}$$

  - $c'(q) > 0$ (because $f(z)$ is increasing)
  - $x''(q) < 0$ (because $f(z)$ is concave)
  - $c(0) = 0$ (we usually assume)



- We can also write producer profits as:

$$\pi(q) = pq - c(q) \tag{6}$$

- To find the value of q that maximizes profits:

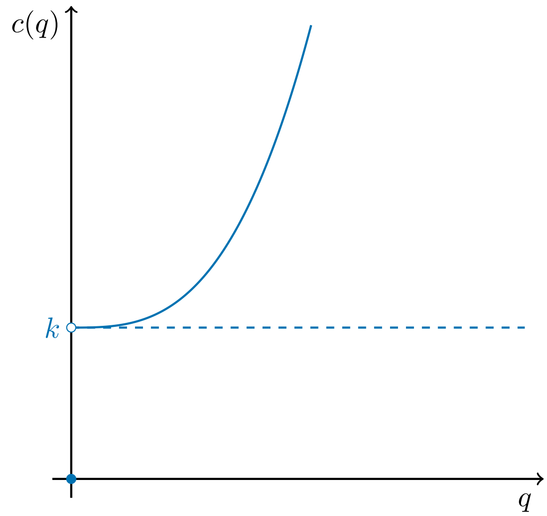
$$\frac{\partial \pi(q)}{\partial q} = 0 \Leftrightarrow p = c'(q) \tag{7}$$

  which yields the oft-mentioned result that for a price-taking producer (who maximizes profits), price equals marginal cost

- Thus, we can solve for the optimal z or the optimal q (given an objective for the producer, for example profit maximization)

10.3.1 Fixed Costs

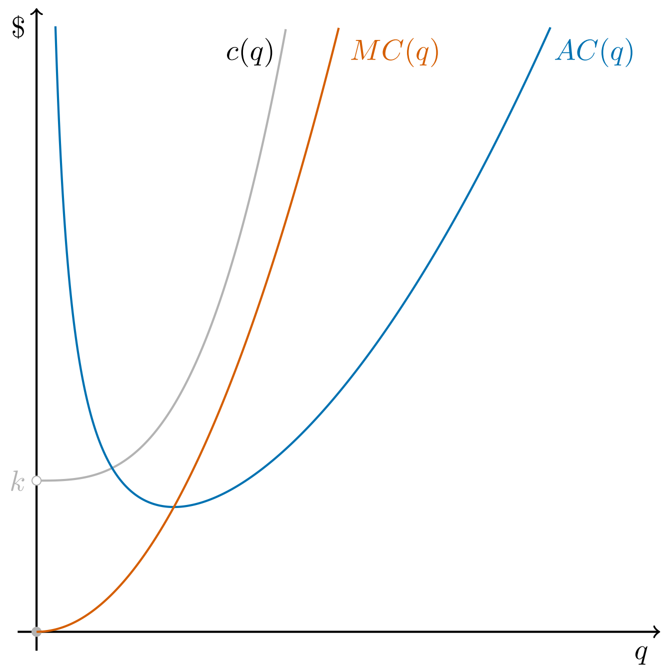
- Adding fixed costs $k > 0$:



- Cost function with fixed costs, $k > 0$:

$$c(q) = \begin{cases} k + wf^{-1}(q) & \text{if } q > 0 \\ 0 & \text{if } q = 0 \end{cases} \tag{8}$$

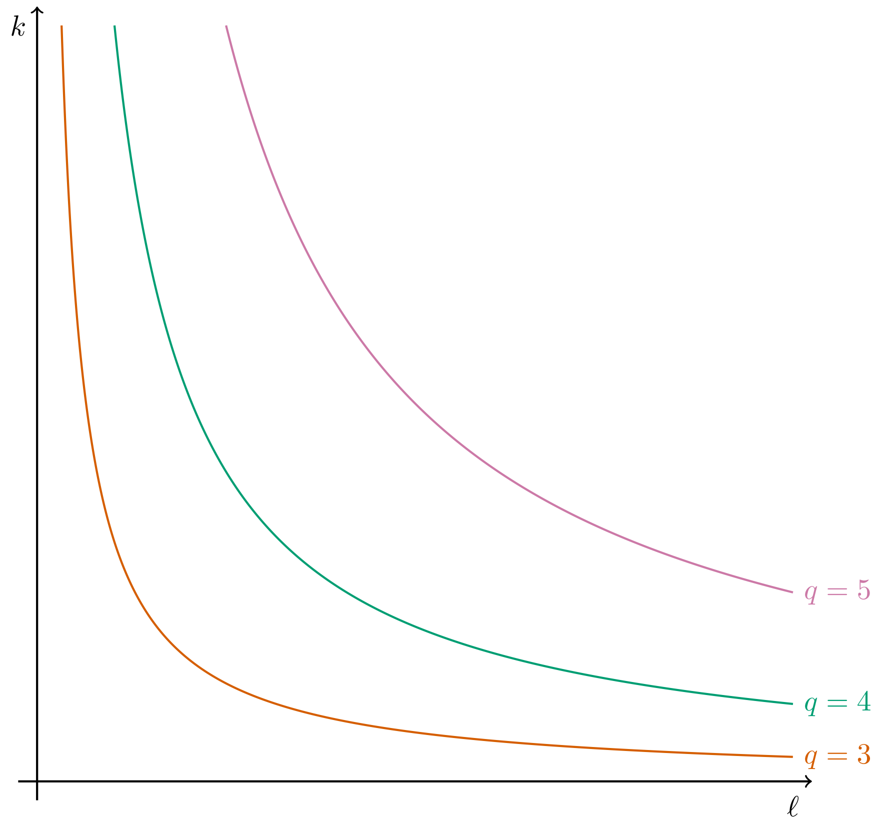
- Since k is independent of q, marginal costs do not depend on k
- As a result, the profit-maximizing level of output is also independent of k
- Total profit does depend on k, and when k is sufficiently large, a firm may lose money on every unit it produces
- For a producer with positive fixed costs, the profit-maximizing level of output does not depend on k, but the producer might prefer to cease operations if p does not exceed **average costs**



- **Claim.** Average costs are increasing if and only if average costs are less than marginal costs
- *Discuss proof in class*

10.4 Multiple Inputs

- Suppose output depends on capital and labor: $q = f(k, \ell)$



- Now the producer needs to choose how much to produce, and also what combination of inputs to use; for every possible level of output (some of which are represented as **isoquants** in the figure above), there are infinitely many combinations of inputs that the producer could use

- We can still set up and solve the producer's profit-maximization problem, which will tell us the profit-maximizing level of each output given the two input prices and the output price

- **Example: Cobb-Douglass Production Function**

$$f(k, \ell) = k^{\frac{1}{4}} \ell^{\frac{1}{4}} \tag{9}$$

  Given this production function, profits are

$$\pi(k, \ell) = p k^{\frac{1}{4}} \ell^{\frac{1}{4}} - rk - w\ell \tag{10}$$

where r represents the rental price of a unit of capital and w is the wage paid for one unit of labor. If the producer maximizes profits, their optimal choice of k and ℓ will solve

$$\frac{\partial \pi(k,\ell)}{\partial k} = 0 \Leftrightarrow \frac{p}{4} k^{-\frac{3}{4}} \ell^{\frac{1}{4}} = r \tag{11}$$

and

$$\frac{\partial \pi(k,\ell)}{\partial \ell} = 0 \Leftrightarrow \frac{p}{4} k^{\frac{1}{4}} \ell^{-\frac{3}{4}} = w \tag{12}$$

It takes some algebra, but we can show that at the profit-maximizing level of output

$$k^* = \left(\frac{p}{4}\right)^2 \frac{1}{r^{3/2} w^{1/2}} \tag{13}$$

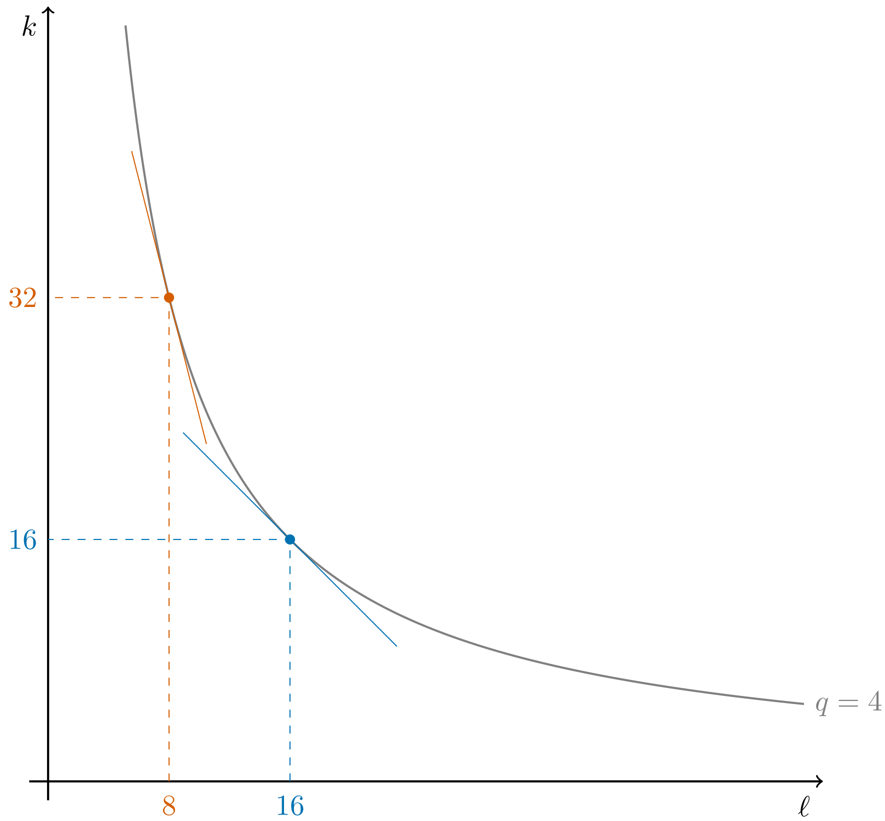
and

$$\ell^* = \left(\frac{p}{4}\right)^2 \frac{1}{r^{1/2} w^{3/2}} \tag{14}$$

Knowing the levels of input use that maximize profits, we can also calculate the profit-maximizing level of production:

$$\begin{aligned} q^* &= k^{\frac{1}{4}} \ell^{\frac{1}{4}} \\ &= \left[\left(\frac{p}{4}\right)^2 \frac{1}{r^{3/2} w^{1/2}}\right]^{\frac{1}{4}} \left[\left(\frac{p}{4}\right)^2 \frac{1}{r^{1/2} w^{3/2}}\right]^{\frac{1}{4}} \\ &= \frac{p}{4} \frac{1}{\sqrt{rw}} \end{aligned} \tag{15}$$

- So, as you can see in the diagram, if the price of the output good were 64 and the price of each input were 4, the producer would produce 4 units of output and use 16 units of capital and 16 units of labor. However, if the wage were $w = 8$ and the rental rate of capital were $k = 2$, the producer would still produce 4 units of output (if $p = 64$), but they'd use 32 units of capital and only 4 units of labor.



- At each of these points, the isoquant is tangent to the line $rk + w\ell = 128$ – but these are two different lines because we are considering two different combinations of input prices

- This illustrates something important: with multiple inputs, the producer is actually solving two different problems. First, they must decide what combination of inputs to use to produce each possible level of output, q; this determines the (minimum) cost of producing q units of output. Then, given an (implicit) cost function for q, the producer chooses their level of production that maximizes profits.

10.5 Cost Minimization

- With multiple inputs, the producer solves two problems:
  - How much to produce?
  - What combination of inputs to use?

- We can separate the problems and solve the latter to the producer's cost function, and then we can use the cost function to find the profit-maximizing level of output
- In our Cobb-Douglas example, the producer first solves:

$$\min_{k,\ell} rk + w\ell \text{ subject to } \underbrace{k^{\frac{1}{4}}\ell^{\frac{1}{4}}}_{=f(k,\ell)} = q \tag{16}$$

- This is a constrained optimization problem that can be represented as a Lagrangian

$$\mathcal{L} = rk + w\ell + \lambda\left(q - k^{\frac{1}{4}}\ell^{\frac{1}{4}}\right) \tag{17}$$

yielding the first-order conditions

$$\frac{\partial \mathcal{L}}{\partial k} = 0 \Leftrightarrow r = \lambda\left(\frac{1}{4}\right)k^{-\frac{3}{4}}\ell^{\frac{1}{4}}, \tag{18}$$

$$\frac{\partial \mathcal{L}}{\partial \ell} = 0 \Leftrightarrow w = \lambda\left(\frac{1}{4}\right)k^{\frac{1}{4}}\ell^{-\frac{3}{4}}, \tag{19}$$

and

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 0 \Leftrightarrow k^{\frac{1}{4}}\ell^{\frac{1}{4}} = q. \tag{20}$$

- Solving these first-order conditions, we arrive at the optimal levels of input use:

$$\ell = q^2\sqrt{\frac{r}{w}} \tag{21}$$

and

$$k = q^2\sqrt{\frac{w}{r}} \tag{22}$$

- As you might expect, an increase in the price of labor causes a decrease in the use of labor and increase in the use of capital, and vice versa

- We can use the levels of input use to characterize the producer's cost function:

$$\begin{aligned} c(q) &= rk + w\ell \\ &= r\left(q^2\sqrt{\frac{w}{r}}\right) + w\left(q^2\sqrt{\frac{r}{w}}\right) \\ &= 2q^2\sqrt{rw} \end{aligned} \tag{23}$$

- Using this cost function in our profit-maximization problem, we confirm that we arrived at the same solution as when we set up the profit-maximization problem as a single step (in the previous section):

$$\begin{aligned} \max_q \pi(q) &= pq - c(q) \\ &= pq - 2q^2\sqrt{rw} \end{aligned} \tag{24}$$

  Yielding the first-order condition:

$$\begin{aligned} \frac{\partial \pi(q)}{\partial q} = 0 &\Leftrightarrow p - 4q\sqrt{rw} = 0 \\ &\Leftrightarrow q^* = \left(\frac{p}{4}\right)\frac{1}{\sqrt{rw}} \end{aligned} \tag{25}$$

  which is the same as before.