5 Utility Maximization

5.1 The Decision-Maker's Problem

- When a decision-maker wants to choose a consumption bundle in $\boldsymbol{R}_{+}^{k}$, we can express their problem as:

$$\max u\left(x_1, x_2, \ldots, x_k\right) \text{subject to} \sum_{i=1}^{k} p_i x_i \leq m \tag{1}$$

  In other word's the decision-maker's problem is to maximize their utility subject to their budget constraint

- When an individual has monotone preferences, we know that they can increase their utility by consuming more of one or both goods; this means that a decision-maker will always use their entire budget, and the budget constraint will bind

- In the two-good case, if we want to solve for the optimal (i.e. chosen or demanded) levels of x_1 and x_2 that a decision-maker will consume, we'll need at least one additional equation to pin down our two unknowns

- **Example: Perfect Complements Utility**
  - Perfect complements: $u(x_1, x_2) = \min\{x_1, x_2\}$
  - *Draw the indifference curves and a budget line*
  - There are infinitely many indifference curves that cross the budget line twice, exactly one indifference curve that touches the budget line once, and infinitely many indifference curves that never intersect the budget line
  - The optimal bundle is where $x_1 = x_2$ and the decision-maker uses her entire budget
  - This gives us two equations and two unknowns, so we can solve for the optimal x_1^* as a function of p_1, p_2, and m, but not x_2
  - Combining these two equations gives us:

$$\begin{aligned} p_1 x_1 + p_2 x_2 = m &\Leftrightarrow p_1 x_1 + p_2 x_1 = m \\ &\Leftrightarrow x_1^*\left(p_1, p_2, m\right) = \frac{m}{p_1 + p_2} \end{aligned} \tag{2}$$

- **Example: Cobb-Douglas Utility**
  - Perfect complements: $u(x_1, x_2) = \ln x_1 + \ln x_2$
  - *Draw the indifference curves and a budget line*
  - Cobb-Douglas indifference curves never intersect the budget line – why?
  - The optimal bundle is at the point of tangency between the budget line and the indifference curve, i.e. where they have the same slope:

$$-\frac{p_1}{p_2} = -\frac{x_2}{x_1} \Leftrightarrow p_1 x_1 = p_2 x_2 \tag{3}$$

    because the slope of the budget line is $-p_1/p_2$ and the slope of the indifference curve is -1 times the marginal rate of substitution (MRS)
  - We can solve for the optimal bundle by using the tangency equation as our second equation, which we can then substitute into the budget constraint:

$$\begin{aligned} p_1 x_1 + p_2 x_2 = m &\Leftrightarrow p_1 x_1 + p_1 x_1 = m \\ &\Leftrightarrow x_1^* (p_1, p_2, m) = \frac{m}{2p_1} \end{aligned} \tag{4}$$

- We've now seen two different ways to solve the utility maximization problem:
  1. Stare at a graph of the indifference curves until insight strikes
  2. Use the tangency condition and the budget constraint as our two equations, which we can solve for our two unknowns $x_1^* (p_1, p_2, m)$ and $x_2^* (p_1, p_2, m)$
- These approaches work in different situations: notice that in the perfect complements example, the indifference curve is not tangent to the budget line at the chosen bundle (or anywhere, in fact)
- With our Cobb-Douglas example, there are two other approaches we might have used to arrive at the same answer
- **Example: Cobb-Douglas Utility revisited, substitution approach**
  - Since we know the budget constraint is binding, we also know that

$$p_1 x_1 + p_2 x_2 = m \Leftrightarrow x_2 = \frac{m - p_1 x_1}{p_2} \tag{5}$$

- We can substitute this into the utility function, so that utility becomes a function of only one choice variable (since whatever money one doesn't spend on Good 1 is spent on good 2):

$$\begin{aligned} u(x_1, x_2) &= \ln x_1 + \ln x_2 \\ &= \ln x_1 + \ln\left(\frac{m - p_1 x_1}{p_2}\right) \\ &= \ln x_1 + \ln(m - p_1 x_1) - \ln p_2 \\ &= u(x_1) \end{aligned} \tag{6}$$

- Now that we have transformed utility into a function of one variable, we can find it's critical values by taking the derivative with respect to x_1 and setting it equal to 0:

$$\begin{aligned} \frac{du(x_1, x_2)}{dlx_1} = 0 &\Leftrightarrow \frac{1}{x_1} - \frac{-p_1}{m - p_1 x_1} = 0 \\ &\Leftrightarrow \frac{1}{x_1} - \frac{-p_1}{m - p_1 x_1} \\ &\Leftrightarrow m - p_1 x_1 = p_1 x_1 \\ &\Leftrightarrow x_1^*(p_1, p_2, m) = \frac{m}{2p_1} \end{aligned} \tag{7}$$

- So, we've again arrived at our expression for demand for Good 1

◦ A final approach to solving the decision-maker's utility maximization problem (or any constrained optimization problem) is to use a **Lagrangian**, and approach which is described in Carl Simon and Lawrence Blume's *Mathematics for Economists* as "truly magical"

- For any constrained optimization problem where we wish to maximize a function $f(x_1, x_2, \ldots, x_k)$ subject to the constraint $g(x_1, x_2, \ldots, x_k) = 0$, we can write the Lagrangian:

$$\mathcal{L} = f(x_1, x_2, \ldots, x_k) + \lambda\left[g(x_1, x_2, \ldots, x_k)\right] \tag{8}$$

where $\lambda \in \boldsymbol{R}$ is the **Lagrange multiplier**

- Instead of finding the derivatives of $f(x_1, x_2, \ldots, x_k)$ with respect to x_1, x_2, etc., we find the derivatives of $\mathcal{L}$ with respect to x_1, x_2, etc. as well as λ

- Setting all of those derivatives equal to zero and solving the system of equations gives us the solution to our constrained optimization problem
- In the case of a utility maximization problem with two goods and a standard budget constraint $p_1x_1 + p_2x_2 = m$, the Lagrangian is:

$$\mathcal{L} = u(x_1, x_2) + \lambda (m - p_1x_1 - p_2x_2) \tag{9}$$

- We then find three derivatives – $\partial\mathcal{L}/partialx_1$, $\partial\mathcal{L}/partialx_2$, and $\partial\mathcal{L}/partial\lambda$
- Setting these three derivatives equal to zero and solving will give us the demand functions that solve the utility maximization problem

◦ **Example: Cobb-Douglas Utility revisited again, Lagrangian approach**

- An individual wishes to maximized $u(x_1, x_2) = \ln x_1 + \ln x_2$ subject to the budget constraint $p_1x_1 + p_2x_2 = m$
- We can write the Lagrangian:

$$\mathcal{L} = \ln x_1 + \ln x_2 + \lambda (m - p_1x_1 - p_2x_2) \tag{10}$$

- Our three first-order conditions are:

(1) $\frac{\partial\mathcal{L}}{\partial x_1} = 0 \Leftrightarrow \frac{1}{x_1} - \lambda p_1 = 0$
(2) $\frac{\partial\mathcal{L}}{\partial x_2} = 0 \Leftrightarrow \frac{1}{x_2} - \lambda p_2 = 0$
(3) $\frac{\partial\mathcal{L}}{\partial\lambda} = 0 \Leftrightarrow m - p_1x_1 - p_2x_2 = 0$

- Solving (1) and (2) for λ and setting the resulting expressions equal to each other yields:

$$\begin{aligned} \frac{1}{p_1x_1} &= \frac{1}{p_2x_2} \\ \Leftrightarrow p_1x_1 &= p_2x_2 \end{aligned} \tag{11}$$

- Notice that this is the same result we got from the tangency condition in our first approach to solving the utility maximization problem given a Cobb-Douglas utility function

- Substituting this into our budget constraint allows us to solve for the demand function (again)

- We've now seen four approaches to solving the utility maximization problem:

  1. Stare at a graph of the indifference curves until insight strikes
  2. Use the tangency condition and the budget constraint as our two equations, which we can solve for our two unknowns $x_1^*(p_1, p_2, m)$ and $x_2^*(p_1, p_2, m)$
  3. Solve the budget constraint for x_2 expressed in terms of x_1, substitute into the utility function, and find the value of $x_1^*(p_1, p_2, m)$ that maximizes utility
  4. Represent the constrained utility maximization problem as the problem of maximizing a(n unconstrained) Lagrangian function

- Approach 1 is usually needed when the utility function is not differentiable, or when preferences are not monotone or strictly convex, or if the budget set is not convex

  - **Example.** What would the demand function look when goods are perfect substitutes, so indifference curves are straight lines? How would the decision-maker maximize utility when the budget line is steeper than the indifference curve? What about when the budget line is flatter than the indifference curve?

- The other three approaches work best when the utility function is differentiable, preferences are strongly monotone and strictly convex, and the budget set is convex

- Using a Lagrangian is particularly helpful as the number of goods starts to increase, particularly if the problem is fairly regular, as the example below illustrates

  - **Example.** Consider a retiree who is deciding how to allocate their retirement savings across the 30 additional years that they expect to live, $t = 1, 2, \ldots, 30$. If they consume c_t in period t, their utility in that period is $u(c_t) = \sqrt{c_t}$. Their total utility is the sum of utility across all periods: $U(c_1, c_2, \ldots, c_30) = \sum_{t=1}^{30} \sqrt{c_t}$. The retiree wishes to maximize total utility subject to the budget constraint that they cannot consume more than their total retirement savings: $\sum_{t=1}^{30} c_t = m$.

5.2 Corner Solutions

- In both our Perfect Complements and Cobb-Douglas examples, the decision-maker always chooses to consume positive quantities of both goods – regardless of the price of the goods or the size of the budget

- An **interior solution** occurs when a decision-maker consumes a positive amount of all the goods; a **corner solution** occurs when a decision-maker sets consumption of at least one good to zero

- **Example: Quasilinear Utility**

  - Consider the utility function:

$$u(x_1, x_2) = \ln x_1 + x_2 \tag{12}$$

  - This is a **quasilinear utility function** where Good 2 is a **numeraire** good that enters into utility linearly

    - Any utility function of the form

$$u(x_1, x_2) = f(x_1) + x_2 \tag{13}$$

    is quasilinear, though it is usually the case tat $f(\cdot)$ is some concave function

  - Suppose a decision-maker wishes to maximize $u(x_1, x_2)$ subject to the budget constraint $p_1x_1 + p_2x_2 = m$

  - This problem turns out to be very easy to solve if we substitute x_2 out of the utility function:

$$\begin{aligned} u(x_1, x_2) &= \ln x_1 + x_2 \\ &= \ln x_1 + \frac{m - p_1x_1}{p_2} \\ &= \ln x_1 + \frac{m}{p_2} - \frac{p_1}{p_2}x_1 \\ &= u(x_1) \end{aligned} \tag{14}$$

– Take the derivative with respect to x_1 and set it equal to zero:

$$\begin{aligned}\frac{du(x_1)}{dx_1} = 0 &\Leftrightarrow \frac{1}{x_1} - \frac{p_1}{p_2} = 0 \\ &\Leftrightarrow x_1 = \frac{p_2}{p_1}\end{aligned} \tag{15}$$

– Notice that the optimal quantity of Good 1 that the decision-maker wishes to consume does not depend on the size of the budget, m, or on the amount of Good 2 that they consume

– At this point, we might think that we've solved the decision-maker's utility maximization problem: we've expressed demand for Good 1 as a function of prices that does not depend on x_2, and we know that whatever money is not spent on Good 1 will be spent on Good 2

– Unfortunately, there's a problem: to see this, consider what happens when $p_1 = 1$, $p_2 = 10$, and $m = 6$

  ◦ The optimal quantity of Good 1 is:

$$\begin{aligned}x_1 &= \frac{p_2}{p_1} \\ &= \frac{10}{1} \\ &= 10\end{aligned} \tag{16}$$

  ◦ If the decision-maker spends the rest of her budget on Good 2, her demand for Good 2 is:

$$\begin{aligned}x_2 &= \frac{m - p_1 x_1}{p_2} \\ &= \frac{6 - 1 \cdot 10}{10} \\ &= -\frac{4}{10} \\ &= -0.4\end{aligned} \tag{17}$$

  ◦ But demand for Good 2 can't be negative!

– What's happened here is that the point of tangency between the budget line and the indifference curve occurs when the quantity of Good 2 consumed in negative

    - The slope of the is $-1/10$
    - The MRS is $1/x_1$, so an indifference curve can only be tangent to the budget line if $x_1 = 10$
    - Given a budget size of $m = 6$, if the price of Good 1 is 1, the MRS is $1/6$ when $x_2 = 0$ – so even if the decision-maker spends her entire budget on Good 1, she can't get to the point where her marginal willingness to substitute matches the price ratio (because Good 2 is very expensive)
  - When the optimal interior solution involves a negative quantity of one of the two goods – which is impossible – the best the decision-maker can do is to spend all of their money on the other good
  - In this case, the consumer wants to spend all of their money on Good 1 unless they can afford to consume at least p_2/p_1 units of Good 1
  - With (this) quasilinear utility function, demand for Good 1 takes the following form:

$$x_1^*(p_1, p_2, m) = \begin{cases} \frac{m}{p_1} & \text{if } p_2 > m \\ \frac{p_2}{p_1} & \text{if } p_2 \leq m \end{cases} \tag{18}$$

  - With quasilinear utility, demand will always take this general form: there will be a range of prices and budget sizes where the consumer spends all of their money on the good that is not the numeraire, and a range of prices and budget sizes where the consumer demands their optimal quantity of the non-numeraire good and spends the rest of their money on the numeraire good

Practice Problems

Find demand for Good 1 for each of the following utility functions:

1. Perfect complements: $u(x_1, x_2) = \min\{ax_1, bx_2\}$ for $a > 0$ and $b > 0$
2. Perfect substitutes: $u(x_1, x_2) = ax_1 + bx_2$ for $a > 0$ and $b > 0$
3. Cobb-Douglas: $u(x_1, x_2) = a \ln x_1 + b \ln x_2$ for $a > 0$ and $b > 0$
4. Square root utility: $u(x_1, x_2) = \sqrt{x_1} + \sqrt{x_2}$

5. $u(x_1, x_2) = \frac{-1}{x_1} + \frac{-1}{x_2}$

6. Constant elasticity of substitution (CES) utility: $u(x_1, x_2) = \frac{x_1^\delta}{\delta} + \frac{x_2^\delta}{\delta}$

7. Quasilinear utility 1: $u(x_1, x_2) = a \ln x_1 + bx_2$ for $a > 0$ and $b > 0$

8. Quasilinear utility 2: $u(x_1, x_2) = a\sqrt{x_1} + bx_2$ for $a > 0$ and $b > 0$

9. Optimal bundle utility: $u(x_1, x_2) = -\sqrt{(x_1 - a)^2 + (x_2 - b)^2}$ for $a > 0$ and $b > 0$

10. Stone-Geary utility: $u(x_1, x_2) = a \ln(x_1 - c) + b \ln(x_2 - d)$ for $a > 0$, $b > 0$, $c > 0$, and $d > 0$

11. What would happen if you extended each of the utility functions above to allow for 3 goods? What about k goods?