3 Demand Functions

- A **demand function** is a mathematical model of individual choice: given any choice set, a demand function tells us what a consumer will choose
  - The budget set characterizes which bundles are available for a given budget size m and price vector $p = (p_1, p_2, \ldots, p_k)$
  - The demand function tells us which affordable bundle will be consumed

3.1 A Formal Definition and Some Examples

- More formally: a **Marshallian demand function** indicates how much of each Good i is chosen as a function of $p = (p_1, p_2, \ldots, p_k)$ and m
  - May also see the Marshallian demand function referred to as Walrasian demand or ordinary demand, or just individual demand (but **not** Hicksian demand)
  - In the two good case, demand is:

$$x(m, p_1, p_2) = \begin{bmatrix} x_1(m, p_1, p_2) \\ x_2(m, p_1, p_2) \end{bmatrix} \tag{1}$$

  - The commodity-specific demand function $x_i(m, p_1, \ldots, p_k)$ tells us demand for Good i (as a function of m and the prices of all the goods)

- **Example 1:** consider a Williams student dividing her time between ski practice and econ homework
  - Every practice run down the ski slope takes $p_{ski} > 0$ hours and every problem set question takes $p_{hw} > 0$ hours
  - She has $m > 0$ hours to divide between skiing and homework
  - What is her budget line?

$$p_{ski}x_{ski} + p_{hw}x_{hw} = m \tag{2}$$

  - Three possible demand functions:

1. Equal numbers of ski runs, homework problems: $x_{ski} = x_{hw}$

$$\begin{aligned} p_{ski}x_{ski} + p_{hw}x_{hw} &= m \\ \Leftrightarrow p_{ski}x_{ski} + p_{hw}x_{ski} &= m \\ \Leftrightarrow x_{ski}\left(p_{ski} + p_{hw}\right) &= m \\ \Leftrightarrow x_{ski} &= \frac{m}{p_{ski} + p_{hw}} \end{aligned} \tag{3}$$

   Since $x_{ski} = x_{hw}$, we have also calculated x_{hw}

2. Equal time spent on skiing, homework: $p_{ski}x_{ski} = p_{hw}x_{hw}$

$$\begin{aligned} p_{ski}x_{ski} + p_{hw}x_{hw} &= m \\ \Leftrightarrow p_{ski}x_{ski} + p_{ski}x_{ski} &= m \\ \Leftrightarrow 2p_{ski}x_{ski} &= m \\ \Leftrightarrow x_{ski} &= \frac{m}{2p_{ski}} \end{aligned} \tag{4}$$

   Since $p_{ski}x_{ski} = p_{hw}x_{hw}$, we know:

$$\begin{aligned} p_{ski}\left(\frac{m}{2p_{ski}}\right) &= p_{hw}x_{hw} \\ \Leftrightarrow \frac{m}{2} &= p_{hw}x_{hw} \\ \Leftrightarrow x_{hw} &= \frac{m}{2p_{hw}} \end{aligned} \tag{5}$$

3. Do 4 runs down the ski slope, then do homework:
   - Student may or may not have enough time (m) to do 4 ski runs
   - **Case 1:** $4p_{ski} \geq m$, all time spent skiing

$$x_{ski} = \frac{m}{p_{ski}} \tag{6}$$

   - **Case 2:** $4p_{ski}Nm$, time left after skiing

$$x_{ski} = 4 \tag{7}$$

     and we can plug x_{ski} into the budget equation to find x_{hw}

- A few things to note about the examples above:
  - → Note that $x_{ski} = x_{hw}$ is not a demand function! A demand function tells us x_i as a function of m and p, **but not as a function of x_i or x_{-i}**
  - → We need to substitute our condition $x_{ski} = x_{hw}$ into our equation for the budget line so that we can work out demand for Good 1 or demand for Good 2 (or both)
  - → Yes, that means we're assuming that our consumer will demand a bundle on the budget line and not in the interior of the budget set – we'll come back to that shortly
  - → You don't always have to write $x_1(m, p_1, p_2)$ when you are doing algebra (we didn't!), but keep in mind the distinction between x_1 as a possible quantity of Good 1 that one might consume and the demand function $x_1(m, p_1, p_2)$

3.2 Properties of Demand Functions

- A demand function satisfies the **adding up property** (also known as Walras' Law) if for all p and all m,

$$\sum_{i=1}^{k} p_i x_i \left(m, p_1, \ldots, p_k\right) = m \tag{8}$$

- In the two-good case, this is equivalent to:

$$p_1 x_1 \left(m, p_1 p_2\right) + p_2 x_2 \left(m, p_1 p_2\right) = m \Leftrightarrow x_2 \left(m, p_1 p_2\right) = \frac{m - p_1 x_1 \left(m, p_1 p_2\right)}{p_2} \tag{9}$$

- The adding up property tells us that the chosen bundle is always on the budget line, but notice that the adding up equation is distinct from the budget equation – one is about points, and one is about functions characterizing chosen bundles
- The implication is that we can differentiate both sides of the adding up equation, because the functional relationship holds for all values of p and m (essentially, both sides of the equation are functions, and they are equal for all values of their arguments)
- Differentiating with respect to m gives us:

$$p_1 \frac{\partial x_1 \left(m, p_1 p_2\right)}{\partial m} + p_2 \frac{\partial x_2 \left(m, p_1 p_2\right)}{\partial m} = 1 \tag{10}$$

- Differentiating with respect to p_1 gives us:

$$\underbrace{p_1 \frac{\partial x_1 (m, p_1 p_2)}{\partial p_1} + x_1 (m, p_1 p_2)}_{\text{by the product rule}} + p_2 \frac{\partial x_2 (m, p_1 p_2)}{\partial p_1} = 0 \quad (11)$$

- A demand function is **homogeneous of degree zero** if for all i

$$x_i (m, p_1, \ldots, p_k) = x_i (\delta m, \delta p_1, \ldots, \delta p_k) \quad (12)$$

for all $\delta > 0$

- Conceptually, when a demand function is homogeneous of degree zero, the units don't matter: multiplying all prices and the budget size by a positive number doesn't change the chosen bundle; so we can move from dollars to cents to euros without changing demand, as long as the set of bundles in the budget set doesn't change

3.3 Responses to Changes in Income

- A good i is a **normal good** if $\frac{\partial x_i (m, p_1, \ldots, p_k)}{\partial m} \geq 0$
- A good that is not a normal good is an **inferior good**, i.e. if $\frac{\partial x_i (m, p_1, \ldots, p_k)}{\partial m} < 0$
- Intuitively, the amount of a good that we demand should not decrease as we get wealthier (i.e. as the size of our budget increases)
- We use the terms normal good and inferior good, but these are properties of demand (i.e. behavior), and even for a specific demand function whether a good is normal or inferior may vary across different values of m and p
- An **Engel curve** is a plot of the relationship between commodity-specific demand, $x_i (m, p_1, \ldots, p_k)$ and m, holding the price vector constant
- We can also express the the relationship between commodity-specific demand and the size of the budget in terms of an **income elasticity**, which tells us the percent change in demand that results from a percent change in m:

$$\varepsilon_i^I = \frac{\partial x_i (m, p_1, \ldots, p_k)}{\partial m} \cdot \frac{m}{x_i (m, p_1, \ldots, p_k)} \quad (13)$$

- Notice that $m > 0$ and $x_i(m, p_1, \ldots, p_k) \geq 0$, so $\varepsilon_i^I < 0$ for inferior goods

3.4 Responses to Price Changes

- The **Law of Demand** states that demand slopes down: an increase in the price of a good leads to a decrease in demand for it
- This is equivalent to assuming that $\frac{\partial x_i(m,p_1,\ldots,p_k)}{\partial p_i} \leq 0$
- A good is a **Giffen good** if $\frac{\partial x_i(m,p_1,\ldots,p_k)}{\partial p_i} > 0$
- We don't just *spend more* on a Giffen good as the price goes up, we actually consume more units of the good (so $x_i(m, p_1, \ldots, p_k)$ increases, and not just $p_i x_i(m, p_1, \ldots, p_k)$)
- We'll also see that a Giffen good must be an inferior good
- We can also characterize Giffen goods in terms of the **own-price elasticity**:

$$\varepsilon_i^{p_i} = \frac{\partial x_i(m, p_1, \ldots, p_k)}{\partial p_i} \cdot \frac{p_i}{x_i(m, p_1, \ldots, p_k)} \tag{14}$$

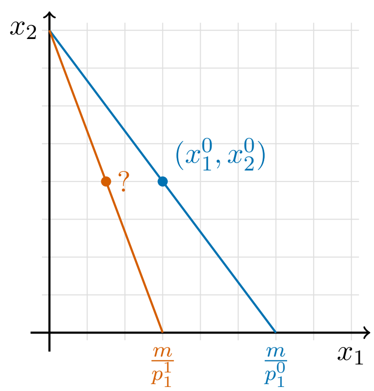
- For a Giffen good, own-price elasticity is positive
- Two goods are (gross) **substitutes** if $\frac{\partial x_i(m,p_1,\ldots,p_k)}{\partial p_{j \neq i}} > 0$
  - When the price of Good j increases, you buy more of Good i (instead)
- Two goods are (gross) **complements** if $\frac{\partial x_i(m,p_1,\ldots,p_k)}{\partial p_{j \neq i}} < 0$
- Once again, we're using the terms complements and substitutes to describe goods, and there are some cases where this may be appropriate (left and right shoes, blue and black pens), but we are talking about a property of a demand function and not of the goods themselves
  - I might think that coffee and milk are complements and someone else might think they are substitutes

3.5 Income and Substitution Effects

- A shift from price $p^0 = (p_1^0, p_2^0)$ to price $p^1 = (p_1^1, p_2^1)$ may lead to a change in demand
  - p^0 = the old price vector
  - p^1 = the new price vector
  - Focus on a change in the price of Good 1 (and impacts on demand for Good 1)
- The **total change in demand** is:

$$\Delta x_1 = x_1\left(p_1^1, p_2^1\right) - x_1\left(p_1^0, p_2^0\right) \tag{15}$$

  - Δx_1 is usually negative if $p_1^1 > p_1^0$ (law of demand)
- What happens when p_1 changes?



- An increase in p_1 has two effcts:
  - Good 1 gets more expensive (in terms of forgone units of Good 2)
  - Budget set shrinks: consumer is poorer in real terms
- Thought experiment: what if we changed the price ratio (to the new one), but also increased the size of the decision-maker's budget so that they could just afford their original bundle?
  - This is a **compensated price change**

- $m^c = p_1^1 x_1^0 + p_2^1 x_2^0$, the cost of the old bundle at the new prices (draw on graph)
- If the consumer faced this hypothetical budget, what would they do? Would they consume more of Good 1, less of Good 1, or is it impossible to tell?

∘ The **substitution effect** is the change in demand for a good that results from a compensated price change: $x_1(p_1^1, \bar{p}_2, m^c) - x_1(p_1^0, \bar{p}_2, \bar{m})$

∘ The **income effect** is the difference between the total change in demand and the substitution effect: $x_1(p_1^1, \bar{p}_2, \bar{m}) - x_1(p_1^1, \bar{p}_2, m^c))$

∘ **Example: calculating income and substitution effects**

- Consider a very simple demand function:

$$x(m, p_1, p_2) = \begin{bmatrix} 2m/3p_1 \\ m/3p_2 \end{bmatrix} \tag{16}$$

- Suppose $p_1^0 = 3$, $\bar{p}_2 = 3$, and $\bar{m} = 54$
- The price of Good 1 rises to 6: $p_1^0 = 6$
- The first step in calculating the substitution effect or the total change in demand is to find the bundle that the consumer would choose at the initial prices:

$$\begin{aligned} x_1\left(\bar{m}, p_1^0, \bar{p}_2\right) &= \frac{2 \cdot 54}{3 \cdot 3} \\ &= 12 \end{aligned} \tag{17}$$

and

$$\begin{aligned} x_2\left(\bar{m}, p_1^0, \bar{p}_2\right) &= \frac{54}{3 \cdot 3} \\ &= 6 \end{aligned} \tag{18}$$

- We can use this bundle to calculate m^c, the budget size that the consumer would need to be able to afford her original bundle at the new prices:

$$\begin{aligned} m^c &= p_1^1 \left[x_1\left(\bar{m}, p_1^0, \bar{p}_2\right)\right] + \bar{p}_2 \left[x_2\left(\bar{m}, p_1^0, \bar{p}_2\right)\right] \\ &= 6 \cdot 12 + 3 \cdot 6 \\ &= 72 + 18 \\ &= 90 \end{aligned} \tag{19}$$

– If she faced the new prices but had a budget of m^c, how much of Good 1 would she choose to consume?

$$\begin{aligned} x_1\left(m^c, p_1^1, \bar{p}_2\right) &= \frac{2 \cdot 90}{3 \cdot 6} \\ &= 10 \end{aligned} \tag{20}$$

– Now we have what we need to calculate the substitution effect: demand for Good 1 at the original prices and budget size, and demand for Good 1 after a compensated price change:

$$\begin{aligned} SE &= x_1\left(m^c, p_1^1, \bar{p}_2\right) - x_1\left(\bar{m}, p_1^0, \bar{p}_2\right) \\ &= 10 - 12 \\ &= -2 \end{aligned} \tag{21}$$

– To calculate the total change in demand, we need demand for Good 1 at the new prices (given the actual size of the budget, $\bar{m}$, as opposed to the budget required for a compensated price change, m^c):

$$\begin{aligned} x_1\left(\bar{m}, p_1^1, \bar{p}_2\right) &= \frac{2 \cdot 54}{3 \cdot 6} \\ &= 6 \end{aligned} \tag{22}$$

– We can now see that the total change in demand for Good 1 is:

$$\begin{aligned} &= x_1\left(\bar{m}, p_1^1, \bar{p}_2\right) - x_1\left(\bar{m}, p_1^0, \bar{p}_2\right) \\ &= 6 - 12 \\ &= -6 \end{aligned} \tag{23}$$

and, as a result, the income effect is $-6 - (-2) = -4$, the difference between the total change in demand and the substitution effect

3.5.1 Substitution Effects and Giffen Goods

- Good i is a Giffen good $\Leftrightarrow \frac{\partial x_i(p_1,\ldots,p_k,m)}{\partial p_i} > 0$
- In the example we just worked through, an increase in the price of Good 1 led to a decrease in demand for Good 1
  - Substitution effect ≤ 0
  - Income effect ≤ 0
- We'll see that the substitution effect is always weakly negative (in the sense that demand moves in the opposite direction from price: an increase in price leads to a (weak) decline in demand after a compensative price change, and a decrease in price leads to a (weak) increase in demand after a compensated price change)
- For a Giffen good, an increase in price leads to an increase in demand: total change in demand is posiitve
  - The total change in demand is the sum of the income effect and the substitution effect
  - The substitution effect is weakly negative
  - Thus, the income effect must be positive: a decrease in the size of your budget leads to an increase in demand for the good
  - In other words: a Giffen good must be inferior