2 Budget Sets

2.1 Terminology: Characterizing the Feasible Set

- A **consumption bundle** $x \in \boldsymbol{R}^k_+$ is a vector indicating the amount of each of k goods that a consumer consumes:

$$x = (x_1, x_2, \dots, x_k)$$

  where x_1 indicates the amount of good 1 that a consumer consumes (or chooses, or does, or eats, or whatever), x_2 indicates the amount of good 2 that a consumer consumes, x_k indicates the amount of good k that a consumer consumes, etc.

- A consumption bundle is a possible answer to the question: "What will I consume/eat/choose/do/etc?"

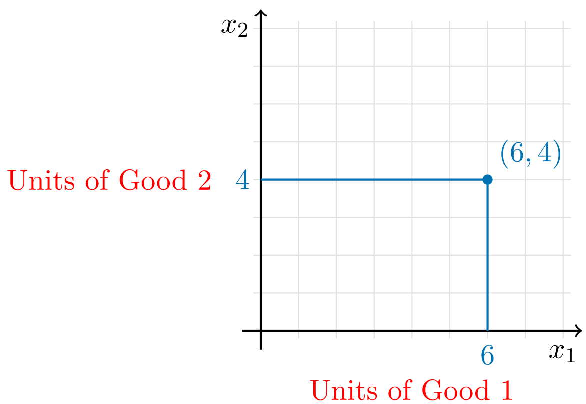
- Some examples:

  1. 2 apples, 3 tomatoes, 6 zucchini
  2. 10 hours of schoolwork, 14 hours of rest
  3. 1 car, 0 motorcycles, 0 boats

- Notice the zeros: we are specifying *all* the goods that we might consume or buy or choose or whatever – there are k of them – and indicating how much of each good we are going to consume.

- We are specifying the scope of our decision problem: we are implicitly saying that what doesn't appear in the consumption bundle is not relevant to the choice problem at hand

- In the special case of $k = 2$, we can plot consumption bundles on a graph. In the graph below, $x_1 = 6$ and $x_2 = 4$.



- A **budget set** is a set of all bundles that an individual can "afford" – or, more generally, the set of all possible feasible things that an individual can choose.
- What do we mean by "afford" in this context?
  - $m > 0$ is an individual's **budget**, which can be denominated in money (income, expenditure, etc.), time, land, or any other scarce resource that an individual needs to decide how to use
  - In the simplest case, every good k has a market price, $p_k \geq 0$, so

$$p = (p_1, p_2, \ldots, p_k)$$

    is a **price vector** characterizing the prices of all the goods – p_1 is the price of Good 1, p_2 is the price of Good 2, and so on
  - Prices are denominated in the same units as the budget (dollars, hours, etc.)
  - Prices don't have to be constant (as they are when the price of good k is just p_k), and they don't have to be explicit market prices; the price is just a measure

of how many units of the scarce budget resource must be used to consume a unit of one of the goods

  – When prices are constant, the cost of purchasing bundle x is

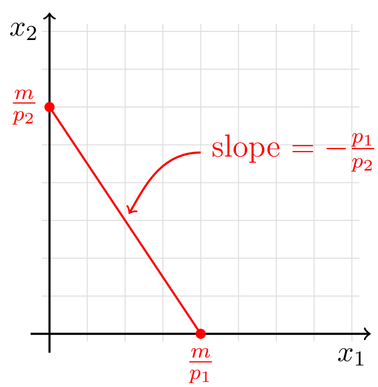
$$p \cdot x = \sum_{i=1}^{k} p_i x_i$$

    which reduces to $p_1 x_1 + p_2 x_2$ in the two-good case

  – A bundle x is affordable if the cost of buying it, given price vector p, is no more than an individual's budget – in other words, a bundle is in the budget set if it is feasible for the individual to choose it, given the size of their budget and the prices of the different goods:

$$\sum_{i=1}^{k} p_i x_i \leq m$$

- When there are only two goods, the set of consumption bundles that cost *exactly* m (given prices p_1 and p_2) can be represented as a line (draw in three steps):

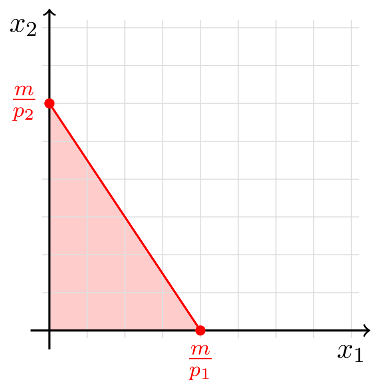


1. If you spend all of your money on Good 1, how much do you consume?
2. If you spend all of your money on Good 2, how much do you consume?
3. The set of consumption bundles that cost exactly m is a **budget line**:

$$p_1 x_1 + p_2 x_2 = m \Leftrightarrow x_2 = \frac{1}{p_2} m - \frac{p_1}{p_2} x_1$$

$\Rightarrow$ The slope of this line is $-\frac{p_1}{p_2}$

- The **budget set** is the set of all bundles that are affordable, in other words, it is the budget line plus every bundle that costs less than m – graphically, the bundles in the triangle made by the budget line and the x and y axes



- What happens to the budget set when you change m? What about p_1? Or p_2?

- When we want to characterize all of the bundles in the budget set defined by m, p_1, and p_2 (or m and $p_1, \ldots, p_k$), we write:

$$\mathbf{B}\left(p_1, p_2, m\right) = \left\{(x_1, x_2) \in \boldsymbol{R}_+^2 : p_1 x_1 + p_2 x_2 \leq m\right\}$$

  This is set notation:

  - What is a set (in Cartesian space): a collection of points?
  - We put our characterization of the set in curly brackets
  - To the left of the colon: a description of what a potential point in the set is, in this case points in (non-negative) Cartesian space
  - The colon is essentially saying "such that"
  - To the right of the colon, we find the condition that characterizes the set – in this case, bundles in Cartesian space that satisfy the budget constraint

2.2 Examples of Budget Sets

2.2.1 Example 1: A Linear Budget Constraint

Professor X has 20 dollars to spend on Spring Street. Lickety Split milkshakes cost five dollars each, and hot cocoas from Tunnel City cost four dollars each. What does her budget set look like?

- What does a consumption bundle look like in this case?

$$x = (x_1, x_2)$$

  where x_1 is the number of milkshakes Professor X consumes and x_2 is the number of hot cocoas she consumes – though the order of the two goods could be reversed without changing anything

- What is her budget constraint?

$$p_1 x_1 + p_2 x_2 \leq m \Leftrightarrow 5x_1 + 4x_2 \leq 20$$

  because the price of milkshakes (x_1) is 5 dollars, the price of hot cocoas (x_2) is four dollars, and her budget is 20 dollars

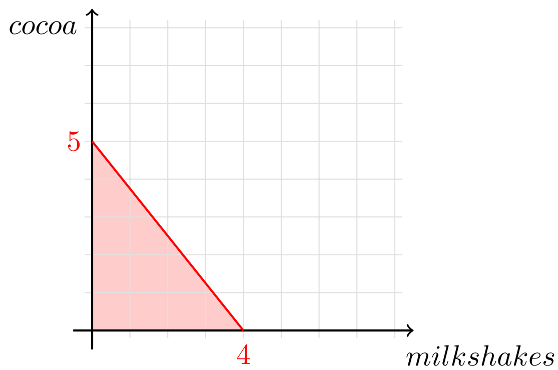
  – The order of the goods doesn't matter – hot cocoa could be Good 1 instead of Good 2 – but it is extremely important to match the price to the corresponding good!

- How would we represent the budget set in set notation?

$$\mathbf{B}\left(5, 4, 20\right) = \left\{(x_1, x_2) \in \boldsymbol{R}_+^2 : 5x_1 + 4x_2 \leq 20\right\}$$

- How would we graph the budget line and the budget set? Rearranging the equation for the budget line yields:

$$\begin{aligned} 5x_1 + 4x_2 = 20 &\Leftrightarrow 4x_2 = 20 - 5x_1 \\ &\Leftrightarrow x_2 = 5 - \frac{5}{4}x_1 \end{aligned}$$



Notice that the slope of the budget line is $-p_1/p_2 = -5/4$.

2.2.2 Example 2: Non-Linear Pricing (Kinked Budget Lines)

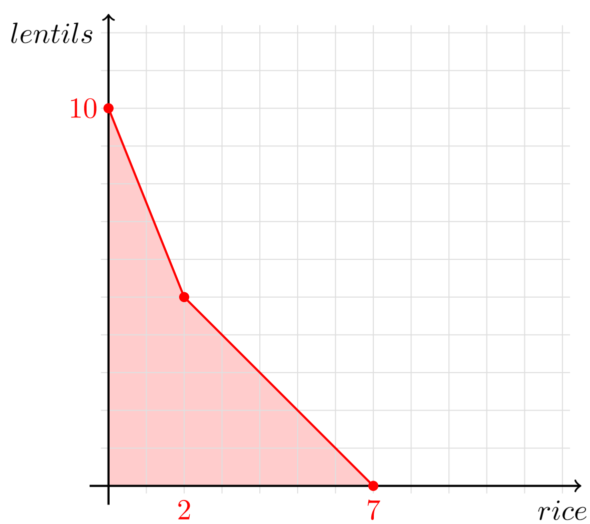
Garret is a vegan who lives on rice and lentils. His income is 100 dollars. Lentils cost 10 dollars per kilogram. The first two one-kilogram bags of rice that one purchases each cost 25 dollars, but all subsequent bags only cost 10 dollars. What does Garret's budget set look like?

- Let's let rice be Good 1 and lentils be Good 2. The slope of the budget line – i.e. the price ratio – is different for $x_1 \leq 2$ vs. $x_2 > 2$:
  - For $x_1 \leq 2$, each bag of rice costs as much as 2.5 bags of lentils
  - For $x_2 > 2$, each bag of rice only costs as much as a bag of lentils
  - This is going to lead to a "kink" in the budget line.
- With a standard budget line, we can start by calculating the endpoints and then connect them with a straight line; with a kinked budget set, we want to calculate the endpoints and the location of the kink
  - To find the y-intercept, we calculate how much of Good 2 (lentils) Garret could buy if he spent his entire budget on lentils:

$$\frac{m}{p_2} = \frac{100}{10} = 10$$

- To find the x-intercept, we calculate how much of Good 1 (rice) Garret could buy if he spent his entire budget on rice – which is harder because the first two bags of rice cost 25 dollars each, but each subsequent bag only costs 10 dollars. The first two kilograms of rice will cost Garret 50 dollars, leaving 50 dollars left over. The remaining 50 dollars is enough to buy 5 kilogram bags of rice at 10 dollars each, so Garret can afford a total of 7 kilograms of rice if he spends his entire budget on rice and nothing on lentils.
- Finally, we can calculate the kink point: if Garret spends 50 dollars buying two bags of rice at 25 dollars each, he has 50 dollars left over – which is enough for five kilograms of lentils.

◦ To graph Garret's budget set, we can plot these three points and connect them with straight lines:



To the left of $x_1 = 2$, the slope of the budget line is $-p_1/p_2 = -2.5$, but to the right of $x_1 = 2$, the slope is much flatter, only -1. Notice that the slope reflects the *relative* price of rice as compared to lentils: we would also observe a budget with an inward kink if the price of lentils decreased after the first five bags. For example, we would have the same budget set if $m = 70$, the price of rice was 10 dollars per kilogram, and the price of lentils was 10 dollars per kilogram for the first five bags, but then four dollars per kilogram thereafter.

- What would we observe if the price of the first few bags of rice were subsidized?

2.2.3 Example 3: Labor-Leisure Tradeoffs and Endowments

Consider a worker dividing her time between work and leisure. She has $L > 0$ total hours in the day. $h \geq 0$ indicates the number of hours worked, $\ell \geq 0$ indicates the number of hours of leisure, and $h + \ell = L$. For every hours worked, our worker receives wage w, so her labor income is wh. She spends her labor income on a consumption good which costs one dollar per unit, so her consumption is $c = wh$. How would we characterize her budget set over consumption and leisure?

2.2.4 Example 4: Time Budgets and Shadow Prices

Super Mom and Super Dad spend their Saturdays doing housework, specifically cooking and cleaning. x_1 indicates the number of rooms cleaned, and x_2 indicates the number of meals cooked. After sleeping, showering, and taking their kids to all of their sporting events, they have $H > 0$ hours left. It takes them x_1^2 hours to clean x_1 rooms, and it takes them x_2^2 hours to cook x_2 meals. How would we characterize Super Mom and Super Dad's time budget?

2.3 Convexity

- We're going to consider three related notions of **convexity**: **convex functions**, **convex combinations**, and **convex sets**

- A real-valued function of one variable is convex if any line connecting two values of the function (secants) lies above the function; for twice differentiable functions of one variable, this is equivalent to having a positive second derivative

- For any two points in k-dimensional Euclidean space, $a = (a_1, a_2, \ldots, a_k)$ and $b = (b_1, b_2, \ldots, b_k)$, a **convex combination** of a and b is a point $c = (c_1, c_2, \ldots, c_k)$ where for $i = 1, 2, \ldots, k$

  $$c_i = \lambda a_i + (1 - \lambda) b_i$$

  for some $\lambda \in [0, 1]$. In other words, a convex combination is a linear combination where all of the linear coefficients sum to one, and none of them is negative.

- **Example:** suppose $a = (a_1, a_2) = (2, 0)$ and $b = (b_1, b_2) = (8, 0)$. A convex combination of a and b is another point $c = (c_1, c_2)$ where

$$\begin{aligned} c_1 &= \lambda a_1 + (1 - \lambda) b_1 \\ &= \lambda * 2 + (1 - \lambda) * 0 \\ &= 2\lambda \end{aligned}$$

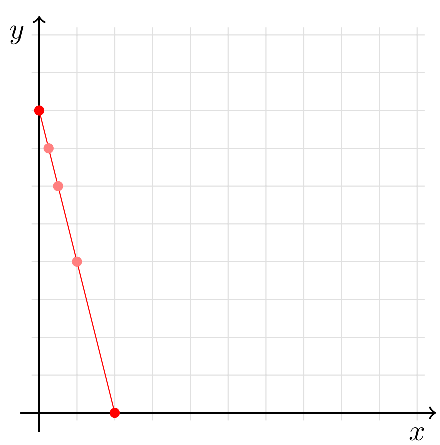
and

$$\begin{aligned} c_2 &= \lambda a_2 + (1 - \lambda) b_2 \\ &= \lambda * 0 + (1 - \lambda) * 8 \\ &= 8 (1 - \lambda) \end{aligned}$$

So, for any $\lambda \in [0, 1]$, $c = (2\lambda, 8(1 - \lambda))$ is a convex combination of $a = (2, 0)$ and $b = (8, 0)$. Some examples:

  - $\lambda = \frac{1}{8}$: $c = \left(\frac{1}{4}, 7\right)$
  - $\lambda = \frac{1}{4}$: $c = \left(\frac{1}{2}, 6\right)$
  - $\lambda = \frac{1}{2}$: $c = (1, 4)$

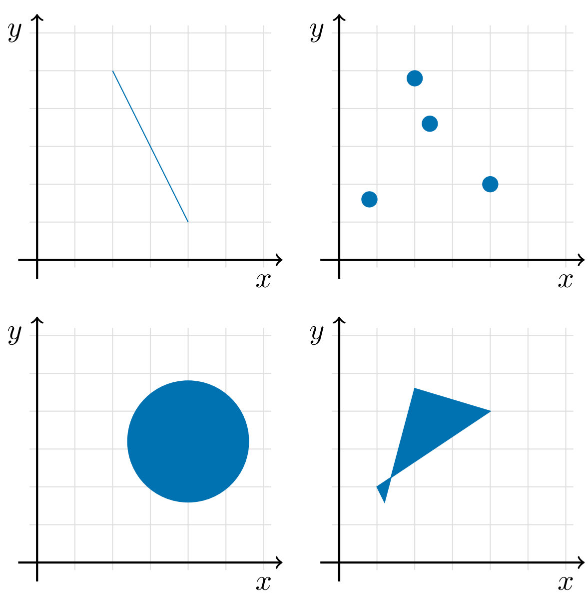
These are all convex combinations of a and b, and when we plot them, we see that they all fall on the line connecting a and b:



- The set of all convex combinations of two points is the line connecting the points
- So, we can go back to our informal definition of a **convex function** and make it more formal: a real-valued function of one variable $f(x)$ is convex ifand only if for every pair of real numbers a and b and every $9 \leq \lambda \leq 1$,

$$\lambda f(a) + (1 - \lambda) f(b) \geq f(\lambda a + (1 - \lambda) b) \tag{1}$$

- Q: What is a **set** in Euclidean space?
  - A: A collection of points (draw some examples)
- Here are some examples of sets in $\boldsymbol{R}^2_+$:

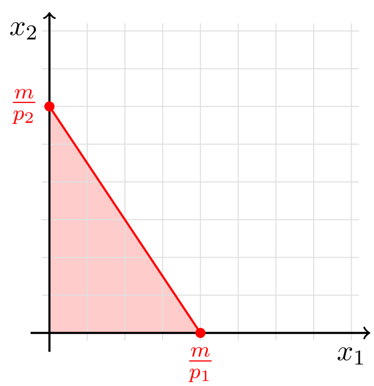


- A set is **convex set** if all convex combinations of points in the set are contained in the set (i.e. if any line connecting two points in the set is also in the set)
  - Which of the sets above is convex?
- We'd like to prove that a linear budget set in $\boldsymbol{R}^2_+$ is a convex set

– Why are we doing this?

1. Whether $\boldsymbol{B}\{p_1, p_2, m\}$ is convex will turn out matter when we solve the utility maximization problem
2. What are we looking for when we ask you to "prove" or "show" something in economics?

– This is an example of a **direct proof** of a statement of the form "If A, then B" – we are going to show that if statement "A" is true (in this case, we have a budget set in $\boldsymbol{R}_+^2$ characterized by a linear budget line $p_1x_1 + p_2x_2 = m$), then statement "B" must also be true (in this case, the budget set is a convex set)



◦ We want to show that the liner budget set

$$\mathbf{B}(p_1, p_2, m) = \left\{(x_1, x_2) \in \boldsymbol{R}_+^2 : p_1x_1 + p_2x_2 \leq m\right\} \tag{2}$$

is a convex set.

◦ By the definition of a convex set, this means we need to show that for any $a \in \mathbf{B}(p_1, p_2, m)$, $b \in \mathbf{B}(p_1, p_2, m)$, and $\lambda \in (0, 1)$, we need to show that the convex combination

$$c = \lambda a + (1 - \lambda)\, b \tag{3}$$

is also in $\mathbf{B}(p_1, p_2, m)$.

◦ We might look at the budget line and say "Duh! It's a triangle." But that is not a formal proof.

- A proof in three steps:

  1. Since $a = (a_1, a_2) \in \mathbf{B}(p_1, p_2, m)$ and $b(b_1, b_2) \in \mathbf{B}(p_1, p_2, m)$, we know that

  $$p_1 a_1 + p_2 a_2 \leq m \tag{4}$$

  and

  $$p_1 b_1 + p_2 b_2 \leq m \tag{5}$$

  2. Since $\lambda \in (0,1)$, and as a result $1-\lambda \in (0,1)$, so we can multiply the inequalities above by λ and $1-\lambda$ (both positive constants), respectively:

  $$\lambda p_1 a_1 + \lambda p_2 a_2 \leq \lambda m \tag{6}$$

  and

  $$(1-\lambda) p_1 b_1 + (1-\lambda) p_2 b_2 \leq (1-\lambda) m \tag{7}$$

  3. We can add two inequalities of the same direction, giving us:

  $$\begin{aligned} \lambda p_1 a_1 + (1-\lambda) p_1 b_1 + \lambda p_2 a_2 + (1-\lambda) p_2 b_2 \leq \lambda m + (1-\lambda) m \\ \Leftrightarrow p_1 [\lambda a_1 + (1-\lambda) b_1] + p_2 [\lambda a_2 + (1-\lambda) b_2] \leq m \\ \Leftrightarrow p_1 c_1 + p_2 c_2 \leq m \end{aligned} \tag{8}$$

  by the definition of c as a convex combination of a and b

This last equation shows that c is in the budget set $\mathbf{B}(p_1, p_2, m)$. QED.